on the segment from the dihedral angle B on one side of plane $x=0$ into the dihedral angle B on the other side of that plane; 3) the trajectory on the segment $[t^{j-1}, t^j]$ belongs, if only partly, to only one angle B on one side of the plane $x=0$; and 4) the trajectory on this segment passes from one dihedral angle B_i to another dihedral angle B_l.

To obtain the lower estimate of I^j, we use the estimates: $\delta(t) \geqslant 0$ when $\gamma \leqslant \beta$: $\delta(t) \geqslant b(1-\cos\beta)$ when $\beta < \gamma < \bar{\gamma}_j$, and $\delta(t) \geqslant b(1-\cos\beta)$ when $\gamma \geqslant \bar{\gamma}_j$, where $\bar{\gamma}_j$ is the root of the equation $\omega_j(\gamma) = 0$. In cases 3) and 4) we use inequality (4.6), and in case 4 - the condition $\beta \leqslant \mu/14$, 4, where μ is the smallest angle between the rays (1.6). Summarizing the estimates I^j over all segments $[t^{j-1}, t^j]$, $j = 1, \ldots, n+1$, we obtain the inequality (2.5) required.

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THE POINCARÉ AND POINCARÉ - CHETAYEV EQUATIONS*

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Poincaré's theory of equations in group variables /1/ has been developed by Chetayev /2/, by his students, and in a number of other investigations. Certain simple observations are made on the Poincaré and Poincaré-Chetayev (PC) equations which should be useful in the application and further study of these equations.

The equations of motion of a mechanical conservative holonomic system with independent coordinates $x_1, \ldots, x_s$ written in the form proposed by Poincaré, have the form

$$\frac{dx_i}{dt} = \xi_i^j(x)\eta_j, \quad i, j, \alpha = 1, \ldots, s \tag{0.1}$$

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \eta_i}\right) = c_{\alpha i}^j \eta_\alpha \frac{\partial L^*}{\partial \eta_j} + X_i L^* \tag{0.2}$$

Here $L^*(x, \eta)$ is the Lagrange function, $\eta_1, \ldots, \eta_s$ are the Poincaré parameters, and repeated indices denote summation. The operators

$$X_i = \xi_j^i(x)\frac{\partial}{\partial x_j} \tag{0.3}$$

form the basis of a certain s-dimensional Lie algebra which we will call algebra A

$$[X_i, X_k] = c_{ik}^\alpha X_\alpha, \quad i, k, \alpha = 1, \ldots, s \tag{0.4}$$

The structural constants are skew symmetric $(c_{ik}^\alpha - c_{ki}^\alpha)$ and satisfy the Jacobi conditions

$$c_{ik}^\alpha c_{\alpha j}^\beta + c_{kj}^\alpha c_{\alpha i}^\beta + c_{ji}^\alpha c_{\alpha k}^\beta = 0 \tag{0.5}$$

It is assumed that the local group of transformations of the configurational space $\{x_1, \ldots, x_s\}$ corresponding to algebra A is transitive, i.e. the following condition holds at the general position points:

$$\det(\xi_i^j(x)) \neq 0 \tag{0.6}$$

As to the rest, the operators (0.3) are arbitrary, so that for a given mechanical system

*Prikl.Matem.Mekhan., 49, 1, 43-55, 1985 the number of methods of choosing these operators (apart from isomorphism, and the arbitrariness of their coordinate realization) is identical with the number of different s-dimensional Lie algebras. A very large number of such algebras exists (e.g. even for $s = 3$ their set has the power of a one-dimensional continuum). The arbitrariness shown, here, on the one hand, creates difficulties in choosing the algebra adequate for the mechanical system in question, but on the other hand it offers new possibilities provided that the choice has been made successfully.

1. Taking into account the non-conservative forces. Properties of the right-hand sides of the Poincaré equations. The Poincaré equations, unlike many other forms of equations of motion, are used to describe conservative systems. Non-conservative forces are also easily accommodated in these equations, provided that we remember that Eqs. (0.2) can be derived from the Lagrange equations by passing to the quasivelocities $\eta_1, \ldots, \eta_s$, chosen using Eqs.(0.1).

Let a Lagrangian system with s degrees of freedom be acted upon, in addition to potential forces, by non-conservative forces $Q_1, \ldots, Q_s$. Having chosen the algebra A and having denoted by $L^*(t, x, \eta)$ the result of substituting (0.1) into the Lagrangian function $L(t, x, x^{\cdot}) : L^*(t, x, \eta) = L(t, x, \xi^j\eta_j)$, we obtain

$$\frac{\partial L^*}{\partial \eta_i} = \xi_j^i \frac{\partial L}{\partial x_j^{\cdot}}, \qquad \frac{\partial L^*}{\partial x_j} = \frac{\partial L}{\partial x_j} + \eta_\beta \frac{\partial \xi_\alpha^\beta}{\partial x_j} \frac{\partial L}{\partial x_\alpha^{\cdot}}$$

$$i, j, \alpha, \beta = 1, \ldots, s$$

Taking into account the Lagrange equation and commutative relations (0.4), we obtain

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \eta_i}\right) = X_i L^* + c_{\alpha i}^j \eta_\alpha \frac{\partial L^*}{\partial \eta_j} + \xi_j^i Q_j$$

The above equations, which are identical with (0.2) when $Q_1 = \ldots = Q_s = 0$ can be written in the form

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \eta_i}\right) = X_i' L^* + \xi_j^i Q_j, \qquad j, i = 1, \ldots, s \tag{1.1}$$

$$X_i' = X_i + c_{\alpha i}^j \eta_\alpha \frac{\partial}{\partial \eta_j}, \qquad \alpha, i, j = 1, \ldots, s \tag{1.2}$$

The resulting Poincaré equations can also be obtained in quasicoordinates /3/ from the Boltzmann-Hamel equations.

Let us write some of the properties of the operators (1.2).

a) The operators $X_1', \ldots, X_s'$ themselves form a basis of a Lie algebra A' isomorphous with the algebra A. Indeed,

$$[X_i', X_k'] = (X_i'\xi_j^k - X_k'\xi_j^i)\frac{\partial}{\partial x_j} + (c_{\alpha k}^j X_i'\eta_\alpha - c_{\alpha i}^j X_k'\eta_\alpha)\frac{\partial}{\partial \eta_j} =$$

$$c_{ik}^\alpha \xi_j^\alpha \frac{\partial}{\partial x_j} - (c_{k\beta}^j c_{\alpha i}^\beta + c_{\beta i}^j c_{\alpha k}^\beta)\,\eta_\alpha \frac{\partial}{\partial \eta_j}$$

Using the identities (0.5) we obtain

$$[X_i', X_k'] = c_{ik}^\alpha \xi_j^\alpha \frac{\partial}{\partial x_j} + c_{\alpha\beta}^j c_{ik}^\beta \eta_\alpha \frac{\partial}{\partial \eta_j} = c_{ik}^\alpha \left(\xi_j^\alpha \frac{\partial}{\partial x_j} + c_{\beta\alpha}^j \eta_\beta \frac{\partial}{\partial \eta_j}\right) = c_{ik}^\alpha X_\alpha'$$

b) the local group of transformations corresponding to algebra A' is a symmetry group of Eqs.(0.1). Indeed, remembering that the operators $X_i + (\xi_j^i)^{\cdot}\partial/\partial x_i^{\cdot}$ correspond to the transformations of the Lagrange variables, we obtain

$$\frac{d\xi_j^\beta}{dt} - \eta_\gamma \xi_i^\beta \frac{\partial \xi_j^\gamma}{\partial x_i} - \xi_j^\alpha \eta_\gamma c_{\gamma\beta}^\alpha = \eta_\gamma\left(\xi_i^\gamma \frac{\partial \xi_j^\beta}{\partial x_i} - \xi_i^\beta \frac{\partial \xi_j^\gamma}{\partial x_i} - c_{\gamma\beta}^k \xi_j^k\right) = 0$$

c) An arbitrarily chosen system of s operators acting in s-dimensional space, for which only condition (0.6) holds, does not form a Lie algebra but is closed, i.e.

$$[X_i, X_k] = a_{ik}^\alpha(x)\, X_\alpha$$

The procedure of deriving the Poincaré equation from the Lagrange equations requires the quantities C_{ik}^α to be constant. The equations of motion can be written for a closed system of operators in the form

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \eta_i}\right) = a_{\alpha i}^j(x)\,\eta_\alpha \frac{\partial L^*}{\partial \eta_j} + X_i L^* + \xi_j^i Q_j \tag{1.3}$$

The additional arbitrariness in the choice of the operators can be used to reduce (in a non-unique manner) the kinetic energy of a scleronomic system to a sum of squares $T = \frac{1}{2}\Sigma\eta_k^2$. Equations (1.3) will then take the form

$$\eta_i^{\cdot} = a_{\alpha i}^{j}(x)\,\eta_\alpha\eta_j + \xi_j^{i}\left(-\frac{\partial U}{\partial x_j} + Q_j\right), \quad L^* = T - U \qquad (1.4)$$
$$i, j, \alpha = 1, \ldots, s$$

In a number of mechanical problems the corresponding transformation appears in a unique manner and generates a Lie algebra directly. Thus the kinematic Euler equations (inversion of Eqs.(0.1))

$$\eta_1 = p = \psi^{\cdot}\sin\theta\sin\varphi - \theta^{\cdot}\cos\varphi,\ \eta_2 = q = \psi^{\cdot}\sin\theta\cos\varphi - \theta^{\cdot}\sin\varphi\ \ \eta_3 = r = \psi^{\cdot}\cos\theta + \varphi^{\cdot}$$

lead to the Poincaré equations of motion of a heavy rigid body about a fixed point, identical to the Euler equations for this problem /1/. The corresponding operators (0.3)

$$X_1 = -\sin\varphi\,\mathrm{ctg}\,\theta\frac{\partial}{\partial\varphi} + \frac{\sin\varphi}{\cos\theta}\frac{\partial}{\partial\psi} + \cos\varphi\frac{\partial}{\partial\theta}$$
$$X_2 = -\cos\varphi\,\mathrm{ctg}\,\theta\frac{\partial}{\partial\varphi} + \frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\psi} - \sin\varphi\frac{\partial}{\partial\theta}$$
$$X_3 = \frac{\partial}{\partial\varphi}$$

define the algebra of a group of rotations

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2$$

Mechanical systems with Euclidean configurational space are simpler examples. The corresponding algebras are commutative.

2. Poincaré-Chetayev equations. Let $x_1, \ldots, x_n$ be the coordinates of a mechanical system with $n - s$ kinematic constraints parametrized by the Poincaré variables $\eta_1, \ldots, \eta_s$

$$x_i^{\cdot} = \xi_i^{j}(x)\eta_j, \quad j = 1, \ldots, s, \quad i = 1, \ldots, n \qquad (2.1)$$

We shall assume that the parametrization (2.1) generates an s-dimensional Lie algebra with the basis

$$X_1 = \xi_i^{1}(x)\frac{\partial}{\partial x_i}, \ldots, \quad X_s = \xi_i^{s}(x)\frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n \quad (s < n) \qquad (2.2)$$

The possible translations of the system are defined in terms of the idependent parameters $\omega_1, \ldots, \omega_s$

$$\delta x_i = \xi_i^{j}(x)\,\omega_j, \quad i = 1, \ldots, n, \quad j = 1, \ldots, s$$

The general equation of dynamics yields

$$\xi_i^{j}(x)\left[\frac{d}{dt}\left(\frac{\partial L}{\partial x_i^{\cdot}}\right) - \frac{\partial L}{\partial x_i} - Q_i\right] = 0 \qquad (2.3)$$

Passing to the Lagrange function L^* and changing somewhat the arguments of Sect.1, we obtain

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial\eta_j}\right) - X_j'L^* - \xi_i^{j}(x)Q_i = \xi_i^{j}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial x_i^{\cdot}}\right) - \frac{\partial L}{\partial x_i} - Q_i\right] \qquad (2.4)$$

Conditions (2.3) yield the relation

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial\eta_j}\right) = X_j'L^* + \xi_i^{j}Q_i, \quad i = 1, \ldots, n, \quad j = 1, \ldots, s$$

first obtained by Chetayev /2/ under the assumption that there were no non-conservative forces present. The equation is written in reduntant coordinates. Just as in the Poincaré equations, the operators

$$X_j' = X_j + c_{\alpha j}^{k}\eta_\alpha\frac{\partial}{\partial\eta_k}, \quad j, k, \alpha = 1, \ldots, s$$

form a basis of a Lie algebra $[X_i', X_k'] = c_{ik}^{\alpha}X_\alpha'$. Certain well-known equations of dynamics are in fact the PC equations, e.g. the Euler-Poisson equations with

$$x_1 = \gamma_1 = \sin\theta\sin\varphi, \quad x_2 = \gamma_2 = \sin\theta\cos\varphi, \quad x_3 = \gamma_3 = \cos\theta, \quad \eta_1 = p, \quad \eta_2 = q, \quad \eta_3 = r$$

3. Poincaré equations for special type non-holonomic systems. The derivation of the PC equations in Sect.2 resembles the derivation of the equations of motion of the non-holonomic systems /3/. The only formal difference between the PC equations and equations of non-holonomic dynamics is the requirement that the operators (2.2) generate an s-dimensional Lie algebra. This requirement, however brings us at once to the problem of the integrability of the kinematic constraints and hence to the conclusion that the PC equations are unsuitable for non-holonomic systems.

The Poincaré equations were written almost simultaneously with the basic forms of the equations of motion of non-holonomic systems. Notwithstanding the considerable degree of resemblance, both theories were developed independently of each other over a long period of time. Generalized PC equations suitable for both holonomic and non-holonomic systems were obtained in /4/. The property of the procedures mentioned above makes it possible to separate a single type of mechanical systems which can be described by the PC equations without any modifications, irrespective of whether the kinematic constraints are integrable or non-integrable.

Let $x_1, \ldots, x_n$ be the coordinates of the system constrained by $n-s$ perfect, stationary kinematic constraints of the form $a_{ki}x_i^{\cdot} = 0$, and only by those constraints (they can be either holonomic, or non-holonomic); L is the Lagrange function and $Q_1, \ldots, Q_n$ are the active non-conservative forces acting on the system. We shall assume that the constraints can be parametrized by (2.1) is such a manner, that the corresponding s operators (2.2) generate an n-dimensional Lie algebra, for which condition (0.6) is satisfied. We will free the system from the constraints by replacing their actions by the reactions of the constraints $R_1, \ldots, R_n$. Replacing in the Lagrange function the velocities $x_i^{\cdot}$ by the parameters η_i according to the formulas

$$x_i^{\cdot} = \xi_i^1(x)\,\eta_1 + \ldots + \xi_i^s(x)\eta_s + \xi_i^{s+1}(x)\eta_{s+1} + \ldots + \xi_i^n(x)\eta_n \tag{3.1}$$

we shall write n equations of motion of the released system in the form of the Poincaré equations (1.1)

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \eta_i}\right) = X_i'L^* + \xi_j^i(Q_j + R_j), \quad i, j = 1, \ldots, n \tag{3.2}$$

Putting

$$\eta_{s+1} = 0, \ldots, \eta_n = 0 \tag{3.3}$$

we satisfy the constraint equations, Then Eqs.(3.2) and relations (3.3) will describe the motion of the initial system and determine, together with the conditions for the constraints to be ideal, the resulting reactions of the constraints. Thanks to the homogeneity of the constraint equations, the possible translations coincide with the actual translations, and we therefore have

$$\delta x_i = \xi_i^1(x)\,\omega_1 + \ldots + \xi_i^s(x)\,\omega_s \tag{3.4}$$

Then the conditions for the constraints to be ideal, taking Eqs. (3.4) into account, yield

$$\xi_i^1(x) R_i = 0, \ldots, \quad \xi_i^s(x) R_i = 0, \quad i = 1, \ldots, n \tag{3.5}$$

Thus the motion of the system in question is described by the first s Poincaré equations (3.2) with conditions (3.3)

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \eta_k}\right) = X_k'L^* + \xi_j^k Q_j, \quad \eta_{s+1} = 0, \ldots, \eta_n = 0 \tag{3.6}$$

and the s equations (3.5) together with the remaining $n-s$ equations of (3.2)

$$\xi_j^\gamma R_j = \frac{d}{dt}\left(\frac{\partial L^*}{\partial \eta_\gamma}\right) - X_\gamma' L^* - \xi_j^\gamma Q_j, \quad \gamma = s+1, \ldots, n \tag{3.7}$$
$$(\eta_{s+1} = \ldots = \eta_n = 0)$$

enable us, thanks to conditions (0.6), to calculate the constraint reactions $R_1, \ldots, R_n$. We note that the first and last term of (3.6) and of the equations in quasicoordinates given in /3/, are identical.

Example. We shall consider the motion of a plate with an edge running along the inclined plane /3/. Here

$$s = 2, \quad n = 3, \quad L = \frac{1}{2}(x^{\cdot 2} + y^{\cdot 2}) + \frac{1}{2}k^2\varphi^{\cdot 2} + gx\sin\alpha$$
$$y^{\cdot} = x^{\cdot}\,\mathrm{tg}\,\varphi$$

The operators $X_1 = \cos\varphi\,\partial/\partial x + \sin\varphi\,\partial/\partial y$, $X_2 = \partial/\partial\varphi$, generating the three-dimensional Lie algebra

$$[X_2, X_1] = -\sin\varphi\frac{\partial}{\partial x} + \cos\varphi\frac{\partial}{\partial y} \equiv X_3, \quad [X_1, X_3] = 0, \quad [X_3 X_2] = X_1$$

correspond to the parametrization of the constraint equations by means of the relations $x^{\cdot} =$

$\eta_1 \cos\varphi,\ y^{\cdot} = \eta_1 \sin\varphi,\ \varphi^{\cdot} = \eta_2$. According to (3.1) we have

$$x^{\cdot} = \eta_1 \cos\varphi - \eta_3 \sin\varphi,\ y^{\cdot} = \eta_1 \sin\varphi + \eta_3 \cos\varphi,\ \varphi^{\cdot} = \eta_2 \tag{3.8}$$

Further, we obtain

$$L^* = \frac{1}{2}(\eta_1^2 + \eta_3^2) + \frac{1}{2} k^2\eta_2^2 + gx\sin\alpha$$
$$X_1' = \cos\varphi \frac{\partial}{\partial x} + \sin\varphi \frac{\partial}{\partial y} + \eta_2 \frac{\partial}{\partial \eta_3}, \qquad X_2' = \frac{\partial}{\partial \varphi}$$
$$X_3' = -\sin\varphi \frac{\partial}{\partial x} + \cos\varphi \frac{\partial}{\partial y} - \eta_2 \frac{\partial}{\partial \eta_1}$$

The equations of motion $\eta_1^{\cdot} = g\sin\alpha\cos\varphi,\ \eta_2^{\cdot} = 0,\ \eta_3 = 0$ and (3.8) are easily integrated

$$\eta_2 = \omega, \qquad \varphi = \omega t + \varphi_0, \qquad \eta_1 = C + \frac{g\sin\alpha}{\omega}\sin\varphi$$
$$x = x_0 + \frac{C}{\omega}(\sin\varphi - \sin\varphi_0) + \frac{g\sin\alpha}{4\omega^2}(\cos 2\varphi_0 - \cos 2\varphi)$$
$$y = y_0 + \frac{C}{\omega}(\cos\varphi_0 - \cos\varphi) + \frac{g\sin\alpha}{4\omega^2}(\sin 2\varphi_0 - \sin 2\varphi) + \frac{g\sin\alpha}{2\omega} t$$

Under the conditions of the problem

$$R_1 \cos\varphi + R_2 \sin\varphi = 0,\ \ R_3 = 0,\ \ -R_1 \sin\varphi + R_2\cos\varphi = g\sin\alpha\sin\varphi + \eta_1\eta_2$$

we obtain, from equations (3.5), (3.7), the constraint reactions

$$R_1 = -(g\sin\alpha\sin\varphi + \eta_1\eta_2)\sin\varphi, \quad R_2 = (g\sin\alpha\sin\varphi + \eta_1\eta_2)\cos\varphi$$

4. The Poincaré equations in terms of redundant parameters. Let us consider once again a holonomic system with s degrees of freedom, the Lagrange function L and the generalized forces $Q_1, \ldots, Q_s$. We shall assume that s operators (0.3) satisfying condition (0.6) and generating an $n > s$-dimensional Lie algebra, were chosen on the strength of certain arguments. Having determined the Lagrangian velocities from the formulas

$$x_i^{\cdot} = \xi_i^1(x)\,\eta_1 + \ldots + \xi_i^n(x)\,\eta_n, \quad i = 1, \ldots, s \tag{4.1}$$

we can obtain directly, as in Sect.2, n identities (2.4) whose left sides are connected, by virtue of condition (0.6), by $n - s$ linear relations. Just as in the case of the mechanical system in question, the Lagrange equations hold and the variables $x_1, \ldots, x_s;\ \eta_1, \ldots, \eta_n$ will satisfy (4.1) and the n equations

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \eta_j}\right) = X_j' L^* + \xi_i^j Q_i, \quad i = 1, \ldots, s, \quad j = 1, \ldots, n \tag{4.2}$$

of which s are algebraically independent. These s independent equations will describe the behaviour of the variables $x_1, \ldots, x_s;\ \eta_1, \ldots, \eta_s$ for any, arbitrarily specified functions

$$\eta_{s+1} = \eta_{s+1}(t, x), \ldots, \eta_n = \eta_n(t, x) \tag{4.3}$$

(in particular, we can put in the first s equations of (4.2) $\eta_{s+1} = \ldots = \eta_n = 0$). The passage to the reduntant parameters can only be justified when the arbitrariness of the functions (4.3) can be dealt with in a resonable manner.

5. Connection with Noether's theorem. Linear integrals. Let us consider the action of the operators (1.2) forming the right-hand sides of the equations of motion (1.1) in the space of the initial Lagrangian variables $x, x^{\cdot}$

$$X_i' L^* = \frac{d}{dt}\left(\frac{\partial L^*}{\partial \eta_i}\right) - \xi_j^i Q_j = \frac{d}{dt}\left(\xi_j^i \frac{\partial L}{\partial x_j^{\cdot}}\right) - \xi_j^i Q_j = \xi_j^i \frac{\partial L}{\partial x_j} + (\xi_j^i)^{\cdot} \frac{\partial L}{\partial x_j^{\cdot}} \equiv X_i^\circ L$$

Thus the function $X_i' L^*$ is proportional to the total variation of the Lagrange function L acted upon by the local none-parameter group of transformations of the space $\{x, x^{\cdot}\}$

$$\frac{dx_j'}{d\tau} = \xi_j^i(x'), \quad \frac{dx_j'^{\cdot}}{d\tau} = (\xi_j^i(x'))^{\cdot}, \quad x_j'|_{\tau=0} = x_j, \quad x_j'^{\cdot}|_{\tau=0} = x_j^{\cdot}$$

Equations (1.1) can now be written in the form

$$\frac{d}{dt}\left(\frac{\partial L^*}{\partial \eta_i}\right) = X_i^\circ L + \xi_j^i Q_j \tag{5.1}$$

If for some i, e.g. $i = 1$,

$$X_1^\circ L + \xi_j^1 Q_j = 0 \tag{5.2}$$

the equations of motion admit of the first integral $\partial L^*/\partial\eta_1 = \text{const}$.

When there are no non-conservative forces, this represents a special case of the Noether's theorem. If the constraints of the system are scleronomic, then conversely, the existence of a linear integral $\omega = \mu_i x_i^{\cdot} = \text{const}$ will apply the existence of a local, one-parameter group of transformations with the operator X_1^0, for which relation (5.2) will hold.

Indeed, in the case in question $L = {}^1/_2\, a_{ij} x_i^{\cdot} x_j^{\cdot} - U(x)$ and the functions $\xi_1^1, \ldots, \xi_s^1$ can be found from the formulas $a_{ij}\xi_j^1 = \mu_i$. Having additionally defined the operators $X_2, \ldots, X_s$ so as to obtain the algebra A, we construct system (5.1). Since

$$\mu_i x_i^{\cdot} = a_{ij}\xi_j^1 x_i^{\cdot} = \xi_j^1 \frac{\partial L}{\partial x_j^{\cdot}} = \frac{\partial L^*}{\partial \eta_1} = \text{const}$$

the first equation of (5.1) yields (5.2).

The determination of linear integrals traditionally involves the concept of cyclic displacements introduced by Chetayev in /2/. Although the presence of cyclic displacements imposes stricter conditions on the equations than Noether's theorem, it often meets with success.

Let us consider in greater detail condition (5.2) for the case of a rheonomic system moving under the action of positional, gyroscopic and dissipative forces

$$L = \frac{1}{2} a_{ij} x_i^{\cdot} x_j^{\cdot} + a_i x_i^{\cdot} + a_0 - U(x)$$
$$Q_i = Q_{jk}^{(i)} x_j^{\cdot} x_k^{\cdot} + Q_j^{(i)} x_j^{\cdot} + Q^{(i)}$$

Having required that the equation of the type (5.2)

$$\xi_j \frac{\partial L}{\partial x_j} + (\xi_j)^{\cdot} \frac{\partial L}{\partial x_j^{\cdot}} + \xi_j Q_j = 0$$

be satisfied identically in $x_1^{\cdot}, \ldots, x_s^{\cdot}$, we obtain

$$\xi_k \frac{\partial a_{ij}}{\partial x_k} + a_{jk} \frac{\partial \xi_k}{\partial x_i} + a_{ik} \frac{\partial \xi_k}{\partial x_j} + 2\xi_k Q_{ij}^{(k)} = 0 \tag{5.3}$$
$$\xi_k \frac{\partial a_i}{\partial x_k} + a_j \frac{\partial \xi_j}{\partial x_i} + \xi_j Q_i^{(j)} = 0, \quad \xi_k \left(\frac{\partial a_0}{\partial x_k} - \frac{\partial U}{\partial x_k} + Q^{(k)} \right) = 0$$

Conditions (5.3) were obtained by tensor methods in /5/ for the case when $a_i = a_0 = Q_{jk}^{(i)} = Q_j^{(i)} = 0$. Here the first subsystem of (5.3) was reduced to the well-known Killing equations defining the local group of motions of a Riemannian space with the metric $ds^2 = a_{ij} dx_i dx_j$.

System (5.3) is overdefined, and, as a rule, is integrable in specific mechanical cases. It may also have no solutions.

The main difficulty encountered in the problem of linear integrals is that of obtaining the effective conditions of their existence expressed in terms of the Lagrange function and its derivatives. The problem has been solved in /6/ only for $s = 2$ and $s = 3$. More general formulations of the Noether's theorem also enable integrals of a more general type to be obtained /7/.

6. Canonical form of Poincaré's equations.

This form of Poincaré's equations was obtained by Chetayev in /2/. The Chetayev's procedure admits also of non-conservative forces. Let us consider Eqs.(1.1) using the assumptions concerning the mechanical system made in Sect.1. We find the function $H^*(x, y)$ using the formula

$$L^*(x, \eta) = \eta_i y_i - H^*(x, y) \tag{6.1}$$

and assuming that the new parameters y_i are defined by the relations $y_i = \partial L^*/\partial \eta_i$. Varying (6.1) we obtain $\partial L^*/\partial x_i = -\partial H^*/\partial x_i$, $\eta_i = \partial H^*/\partial y_i$, and equations (0.1) and (1.1) will now take the form

$$x_i^{\cdot} = Y_i^* H^*, \quad y_i^{\cdot} = -X_i^* H^* + \xi_j^i Q_j, \quad i, j = 1, \ldots, s \tag{6.2}$$
$$Y_i^* = \xi_i^j(x) \frac{\partial}{\partial y_j}, \quad X_i^* = \xi_j^i(x) \frac{\partial}{\partial x_j} + c_{ij}^{\gamma} y_{\gamma} \frac{\partial}{\partial y_j}, \quad \gamma = 1, \ldots, s \tag{6.3}$$

In (6.1) H^* is the Hamiltonian function:

$$H^*(x, y) = \eta_i y_i - L^*(x, \eta) = \eta_i \frac{\partial L^*}{\partial \eta_i} - L^*(x, \eta) = \xi_j^i \eta_i \frac{\partial L}{\partial x_j^{\cdot}} - L = x_j^{\cdot} p_j - L = H(x, p)$$

Equations (6.2) can also be obtained directly from Hamilton's equations. To do this it is sufficient, having chosen the algebra A (see Sect.1), to carry out in Hamilton's equations $x_i^{\cdot} = \partial H/\partial p_i$, $p_i^{\cdot} = -\partial H/\partial x_i + Q_i$ a linear substitution of the moments using the formulas

$y_k = \xi_i^k(x)p_i$. We must also put $H^* = H(x, \alpha_k y_k)$, $\alpha_k^l \xi_i^k = \delta_i^l$ (δ_i^l is the Kronecker delta.

The operators Y_i^* form the basis of the commutative Lie algebra

$$[Y_i^*, Y_k^*] = 0 \tag{6.4}$$

This follows from the identities $\partial \xi^j_k / \partial y_i \equiv 0$. The operators X_i^* generate a Lie algebra isomorphic with algebra A. Indeed,

$$[X_i^*, X_k^*] = (X_i^* \xi_j^* - X_k^* \xi_j^i) \frac{\partial}{\partial x_j} + (c_{kj}^\gamma X_i^* y_\gamma - c_{ij}^\gamma X_k^* y_\gamma) \frac{\partial}{\partial y_j} =$$
$$c_{ik}^\alpha \xi_j^\alpha \frac{\partial}{\partial x_j} - (c_{jk}^\alpha c_{i\alpha}^\gamma + c_{ij}^\alpha c_{k\alpha}^\gamma) y_\gamma \frac{\partial}{\partial y_j}$$

Using the Jacobi conditions (0.5) we obtain

$$[X_i^*, X_k^*] = c_{ik}^\alpha \xi_j^\alpha \frac{\partial}{\partial x_j} + c_{ik}^\alpha c_{\alpha j}^\gamma y_\gamma \frac{\partial}{\partial y_j} = c_{ik}^\alpha X_\alpha^* \tag{6.5}$$
$$i, k, \alpha = 1, \ldots, s$$

The system of $2s$ operators $Y_1^*, \ldots, Y_s; X_1^*, \ldots, X_s^*$, acting in $2s$-dimensional phase space is closed and has a simple multiplication table defined by the cummutative relations (6.4), (6.5) and

$$[X_i^*, Y_k^*] = \frac{\partial \xi_k^i}{\partial x_j} Y_j^* \tag{6.6}$$

The latter can easily be confirmed

$$[X_i^*, Y_k^*] = (X_i \xi_k^j - Y_k^* c_{ij}^\gamma y_\gamma) \frac{\partial}{\partial y_j} = \left(\xi_\gamma^i \frac{\partial \xi_k^j}{\partial x_\gamma} + c_{ji}^\gamma \xi_k^\gamma\right) \frac{\partial}{\partial y_j} =$$
$$\xi_\gamma^j \frac{\partial \xi_k^i}{\partial x_\gamma} \frac{\partial}{\partial y_j} = \xi_j^\gamma \frac{\partial \xi_k^i}{\partial x_j} \frac{\partial}{\partial y_\gamma} = \frac{\partial \xi_k^i}{\partial x_j} Y_j^*$$

The matrix of the operators (6.3) is (the prime denotes transposition)

$$M = \begin{Vmatrix} 0 & \tau \\ -\tau' & \sigma \end{Vmatrix}, \quad \tau = (\xi_j^i), \quad \sigma = (\zeta_j^i), \quad \zeta_j^i = -c_{ji}^\gamma y_\gamma \tag{6.7}$$

The matrix σ is skew symmetric ($\zeta_j^i = -c_{ij}^\gamma y_\gamma = c_{ji}^\gamma y_\gamma = -\zeta_i^j$), so that when s is odd, we have $\det \sigma = 0$. In addition to σ, all principal diagonal minors of the matrix M are also skew symmetric. Those of odd order are degenerate. By virtue of condition (0.6) $\det M = (\det \tau)^2 \neq 0$.

In case of the usual Hamiltonian system $\xi_j^i = \delta_j^i$, $c_{ij}^\gamma = 0$, $y_i = p_i$, $X_i^* = \partial/\partial x_i$, $Y_i^* = \partial/\partial p_i$ the motion (6.7) takes the form

$$M = \begin{Vmatrix} 0 & E \\ -E & 0 \end{Vmatrix}$$

and the operators (6.3) define a $2s$-dimensional commutative Lie algebra.

7. Operator of displacement along the trajectories of motion. The operator of displacement along the trajectories of motion of an autonomous system of equations $z_i^\cdot = f_i(z)$ will be understood to be the operator of differentiation, with respect to time, of the functions defined in the phase space $\{z\}$:

$$S = \frac{d}{dt} = f_i \frac{\partial}{\partial z_i}$$

Let us consider a mechanical conservative system with stationary constraints described by Eqs.(6.2). The displacement operator of such a system

$$S = Y_i^* H^* \frac{\partial}{\partial x_i} - X_i^* H^* \frac{\partial}{\partial y_i}$$

can be represented in the form

$$S = -\frac{\partial H^*}{\partial x_i} Y_i^* + \frac{\partial H^*}{\partial y_i} X_i^* \tag{7.1}$$

Indeed, using the notation of (6.3) we obtain

$$S = \xi_i{}^j \frac{\partial H^*}{\partial y_j} \frac{\partial}{\partial x_i} - \left(\xi_j{}^i \frac{\partial H^*}{\partial x_j} + c_{ij}{}^\gamma y_\gamma \frac{\partial H^*}{\partial y_j} \right) \frac{\partial}{\partial y_i} = -\frac{\partial H^*}{\partial x_i} \xi_i{}^j \frac{\partial}{\partial y_j} + \frac{\partial H^*}{\partial y_i} \left(\xi_j{}^i \frac{\partial}{\partial x_j} + c_{ij}{}^\gamma y_\gamma \frac{\partial}{\partial y_j} \right) = -\frac{\partial H^*}{\partial x_i} Y_i{}^* + \frac{\partial H^*}{\partial y_i} X_i{}^*$$

Thus the displacement operator S belongs to a linear shell stretched over the operators $Y_i{}^*$, $X_i{}^*$. If the algebra A is chosen in such a manner that

$$\frac{\partial \xi_j{}^i}{\partial x_1} = \ldots = \frac{\partial \xi_j{}^i}{\partial x_l} = 0, \quad i = 1, \ldots, s; \quad j = l+1, \ldots, s \tag{7.2}$$

and

$$\frac{\partial H^*}{\partial x_1} = \ldots = \frac{\partial H^*}{\partial x_l} = 0 \tag{7.3}$$

then

$$S = -\frac{\partial H^*}{\partial x_{l+1}} Y_{l+1}^* - \ldots - \frac{\partial H^*}{\partial x_s} Y_s{}^* + \frac{\partial H^*}{\partial y_i} X_i{}^* \tag{7.4}$$

The system of equations

$$Y_{l+1}{}^*\omega = \ldots = Y_s{}^*\omega = X_1{}^*\omega = \ldots = X_s{}^*\omega = 0$$

is compatible. and everyone of its solutions is a first integral of the equations of motion (6.2). (This follows from the formulas (7.4), (6.4)–(6.6). Such a situation arises, in particular, in the case when $x_1, \ldots, x_l$ are explicit ignorable coordinates.

The properties of the displacement operator S can have various applications. The simplest applications are discussed in Sects.8 and 9.

8. Non-linear first integrals of mechanical systems moving inertially.

Let a mechanical system move inertially and let its kinetic energy depend only on the parameters $y_1, \ldots, y_s$. Then

$$H^* = T(y), \quad S = \frac{\partial H^*}{\partial y_i} X_i{}^*$$

The system of equations

$$X_1{}^*\omega = \ldots = X_s{}^*\omega = 0 \tag{8.1}$$

is consistent and has, according to condition (0.6), exactly s functionally independent solutions. The solutions represent the first integrals of the equations of motion, and at the same time the invariants of the corresponding local groups of transformations. We shall restrict ourselves to the case $s = 3$. We shal seek the first integrals depending on the parameters y_1, y_2, y_3 only. Equations (8.1) will then take the form

$$c_{12}^j y_j \frac{\partial \omega}{\partial y_2} + c_{13}^j y_j \frac{\partial \omega}{\partial y_3} = 0, \quad c_{21}^j y_j \frac{\partial \omega}{\partial y_1} + c_{23}^j y_j \frac{\partial \omega}{\partial y_3} = 0$$
$$c_{31}^j y_j \frac{\partial \omega}{\partial y_1} + c_{32}^j y_j \frac{\partial \omega}{\partial y_2} = 0 \tag{8.2}$$

Since $\det \sigma = 0$, at least one integral of the system exists. If the algebra A is noncummutative, then it will be a unique integral of the form $\omega = \omega(y)$, in every case discussed below.

We shall utilize the results obtained in /8/ where the whole space of structural constants $c_{ij}{}^\gamma$ of the three-dimensional Lie algebra is split by the explicit algebraic conditions into parts corresponding to the mutually non-isomorphic algebras. The three-dimensional algebras split into two series, discrete and continuous. The discrete series contains five algebras (not including the commutative), and the continuous series contains a countless number of algebras.

The algebras of discrete series are separated by the condition

$$(c_{12}{}^2 + c_{13}{}^3)^2 + (c_{13}{}^1 + c_{23}{}^2)^2 + (c_{12}{}^1 - c_{23}{}^3)^2 = 0$$

After integrating Eqs.(8.2), we obtain a family of first integrals

$$\omega = c_{32}{}^1 y_1{}^2 + c_{13}{}^2 y_2{}^2 + c_{21}{}^3 y_3{}^2 + 2c_{32}{}^2 y_1 y_2 + 2c_{32}{}^3 y_1 y_3 + 2c_{13}{}^3 y_2 y_3 \tag{8.3}$$

The local groups corresponding to the algebras of finite series, can be realized in the form of the following groups of transformations: a group of rotations, a group of motions of a Lobachevskii plane in a Euclidean plane, and a group of Lorentz and Galileo transformations of the two-dimensional space-time. In accordance with this the quadratic forms (8.3) differ from each other in their signatures (using a non-degenerate change of variable we can reduce y_i to the corresponding canonical forms: $y_1^2+y_2^2+y_3^2$, $y_1^2+y_2^2-y_3^2$, $y_1^2+y_2^2$, $y_1^2-y_2^2$, y_1^2).

The algebras of the continuous series are separated by the condition

$$(c_{12}^2+c_{13}^3)^2+(c_{13}^1+c_{23}^2)^2+(c_{12}^1-c_{23}^3)^2>0$$

and we can assume, without loss of generality, e.g. that $c_{13}^1+c_{23}^2\neq 0$.

Using the structural constants we calculate explicitly the characteristic parameter c_0, which can take all real values. The non-isomorphic algebras correspond to two different values of the parameter. We will put

$$u_1=y_1-\mu_0 y_3,\quad u_2=y_2-\nu_0 y_3$$

$$\mu_0=\frac{c_{13}^2c_{23}^3-c_{13}^3c_{23}^2}{c_{13}^1c_{23}^2-c_{13}^2c_{23}^1},\quad \nu_0=\frac{c_{13}^3c_{23}^1-c_{13}^1c_{23}^3}{c_{13}^1c_{23}^2-c_{13}^2c_{23}^1}$$

(if $c_{13}^1+c_{23}^2\neq 0$, then the parameters μ_0, ν_0 have finite values).

When $c_0=\alpha^2\ (\alpha>0)$, we obtain the first integral

$$\omega=[2c_{13}^2u_2+(c_{13}^1-c_{23}^2-\alpha(c_{13}^1+c_{23}^2))u_1]^{\alpha+1}\times[2c_{13}^2u_2+(c_{13}^1-c_{23}^2+\alpha(c_{13}^1+c_{23}^2))u_1]^{\alpha-1}$$

and when $c_0=-\alpha^2\ (\alpha>0)$, we have integrals of the form

$$\omega=[c_{13}^2u_2^2+(c_{13}^1-c_{23}^2)u_1u_2-c_{23}^1u_1^2]\times\exp\left\{\frac{2}{\alpha}\operatorname{arctg}\frac{2c_{13}^2u_2+(c_{13}^1-c_{23}^2)u_1}{\alpha(c_{13}^1+c_{23}^2)u_1}\right\}$$

An algebra for which $c_0=0$ and $(c_{23}^1)^2+(c_{13}^2)^2+(c_{12}^3)^2>0$, is singled out as singular (we can assume e.g. without loss of generality that $c_{13}^2\neq 0$). The corresponding first integral can be written in the form

$$\omega=\left(u_2+\frac{c_{13}^1-c_{23}^2}{2c_{13}^2}u_1\right)\exp\left[-\frac{c_{13}^1+c_{23}^2}{2c_{13}^2}\,\frac{u_1}{u_2+(c_{13}^1-c_{23}^2)u_1/(2c_{13}^2)}\right]$$

In the limiting case $c_0=0$, $(c_{23}^1)^2+(c_{13}^2)^2+(c_{12}^3)^2=0$ we also have the corresponding singular algebra of the continuous series and the first integral

$$\omega=\frac{c_{13}^1y_1+c_{13}^3y_3}{c_{13}^1y_2+c_{23}^3y_3}\quad (c_{13}^1\neq 0)$$

We note a characteristic aspect here, that in the Chetayev variables y_1, y_2, y_3 all integrals obtained depend only on the structure of the algebra A.

Detailed discussions of the structure of three-dimensional Lie algebras were instrumental in solving a mechanical problem, i.e. in computing the additional first integrals. The problem can obviously be inverted, so that integration of Eqs.(8.1) will yield the structural characteristics of the Lie algebras of dimension $s>3$.

9. Particular solutions of the equations of motion generated by the non-cyclic first integrals. The linear envelope of the operators (6.3) contains, together with the displacement operator S, the commutators $[S, X_i^*]$, $[S, Y_i^*]$. Indeed, using Eq.(7.1) we obtain

$$[S,\ X_i^*]=\mu_i^jY_j^*+\nu_i^jX_j^*,\quad [S,\ Y_i^*]=\pi_i^jY_j^*+\rho_i^jX_j^* \tag{9.1}$$

$$\mu_i^j=X_i^*\frac{\partial H^*}{\partial x_i}+\frac{\partial\xi_k^i}{\partial x_j}\frac{\partial H^*}{\partial x_k},\quad \nu_i^j=-X_i^*\frac{\partial H^*}{\partial y_j}+c_{ki}^j\frac{\partial H^*}{\partial y_k}$$

$$\pi_i^j=Y_i^*\frac{\partial H^*}{\partial x_j}+\frac{\partial\xi_i^k}{\partial x_j}\frac{\partial H^*}{\partial x_k},\quad \rho_i^j=-Y_i^*\frac{\partial H^*}{\partial y_j}$$

Let ω be the first integral of system (6.2) when $Q_1=\ldots=Q_s=0$: $S\omega=0$. We write

$$X_i^*\ \omega=a_i,\ Y_i^*\omega=b_i \tag{9.2}$$

If the integral ω does not correspond to any explicit ignorable coordinate, then not all functions a_i, b_i will be identically zero. Let us assume that the system of equations

$$a_i=b_i=0,\ i=1,\ldots,s \tag{9.3}$$

is consistent. Then the system will define a particular integral of Eqs.(6.2). This follows directly from (9.1):

$$Sa_i=\mu_i^jb_j+\nu_i^ja_j,\ Sb_i=\pi_i^jb_j+\rho_i^ja_j$$

It should be noted that the assertion concerning the particular integrals can be regarded as a modification of the theorem on stationary motions, supplying an extremum to one of the integrals on the level surfaces of the remaining integrals /9/. The possiblity of obtaining particular solutions can be found useful even in the case of fully integrable problems.

The question of the dimensionality of the manifold (9.3) requires a separate assessment. Here we shall only note that in many mechanical problems the dimension is not less than one. This is explained by the fact that the properties of the generating integrals ω and the matrix (6.7) make it possible to obtain linear identity relations connecting the functions (9.2). Since $S\omega = 0$, one of these relations will always be

$$\frac{\partial H^*}{\partial y_i} a_i - \frac{\partial H^*}{\partial x_i} b_i = 0$$

As an example we shall consider the Euler-Poisson equations for the problem of the motion of a heavy rigid body about a fixed point. These equations represent the PC equations (Sect.2) and are identical at the same time with their canonical form (provided that we return in the latter to the p, q, r variables). Since we have here an ignorable coordinate, it follows that the displacement operator S is a linear combination of five operators only (see (7.4)). A slightly different formulation of this operator is also of interest. The formulation is obtained directly from the representation (7.1)

$$S = pX_1^* + qX_2^* + rX_3^* + Px_c Y_1 + Py_c Y_2 + Pz_c Y_3$$

where

$$X_1^* = \frac{C}{B} r \frac{\partial}{\partial q} - \frac{B}{C} q \frac{\partial}{\partial r} + \gamma_3 \frac{\partial}{\partial \gamma_2} - \gamma_2 \frac{\partial}{\partial \gamma_3}, \qquad Y_1 = \frac{\gamma_3}{B} \frac{\partial}{\partial q} - \frac{\gamma_2}{C} \frac{\partial}{\partial r} \tag{9.4}$$
$((123), (pqr), (ABC))$

The operators (9.4) generate a six-dimensional Lie algebra, and the group corresponding to it can be realized as a group of motions of a three-dimensional Euclidean space.

We have two identities of different origin

$$\gamma_1 Y_1 + \gamma_2 Y_2 + \gamma_3 Y_3 = 0, \ ApY_1 + BqY_2 + CrY_3 + \gamma_1 X_1^* + \gamma_2 X_2^* + \gamma_3 Y_3^* = 0$$

The existence of the first identity is explained by the redundant nature of the variables $\gamma_1, \gamma_2, \gamma_3$, and the second is caused by the presence of an ignorable coordinates. No other linear relations exist. Since the operators (9.4) exist in a six-dimensional space, the system of equations

$$Y_i \omega = X_i^* \omega = 0, \ i = 1, 2, 3$$

is consistent and has exactly two solutions, namely the angular momentum integral relative to the vertical, and the geometrical integral. Consequently, the integrals shown cannot generate any particular solutions. In the Euler case $(x_c = y_c = z_c = 0)$ $S = pX_1^* + qX_2^* + rX_3^*$, and the system of equations $X_1^*\omega = X_2^*\omega = X_3^*\omega = 0$ yields another integral, namely the Euler integral (this is the first integral of (8.3) corresponding to the group of rotations).

Let us list some particular solutions generated by non-cyclic first integrals.

The energy integral

$$H = {}^1/_2(Ap^2 + Bq^2 + Cr^2) + P\,(x_c\gamma_1 + y_c\gamma_2 + z_c\gamma_3)$$

leads to a family of permanent rotations.

In the Euler case the Euler integral generates a particular integral /10/

$$Ap/\gamma_1 = Bq/\gamma_2 = Cr/\gamma_3$$

In the Lagrange case $(x_c = y_c = 0, A = B)$ the integral $\omega = r$ gives permanent rotations of a special type:

$$p = q = 0,\ r = r_0,\ \gamma_1 = \gamma_2 = 0,\ \gamma_3 = \pm 1$$

We will take the generating first integral in the form of a combination of the Lagrange integral and the energy integrate $\omega = {}^1/_2\,A\,(p^2 + q^2) + Pz_c\gamma_3 + \alpha r^2 + \beta r$ where α, β are constant coefficients. System (9.3) reduces to four equations

$$p/\gamma_1 = q/\gamma_2 = \lambda,\ \beta = C\lambda\gamma_3 - 2\alpha r,\ (2\alpha A - C^2)\, r\lambda + \beta A\lambda + Pz_c C = 0$$

Substituting this particular integral into the equations of motion yields a family of regular precessions /10/

$$p = \lambda\gamma_1,\ q = \lambda\gamma_2,\ \gamma_1^{\cdot} = \Omega\gamma_2,\ \gamma_2^{\cdot} = -\Omega\gamma_1,\ \Omega = (A - C)/r_0 A + Pz_c/\lambda A$$
$$\lambda = \text{const},\ r = r_0,\ \gamma_3 = \gamma_3^0 = \text{const},\ A\gamma_3^0\lambda^2 - Cr_0\lambda + Pz_c = 0$$

In the Kovalevskaya case $(y_c = z_c = 0,\ A = B = 2C)$ we shall consider the combination of the Kovalevskaya and the energy integral

$$\omega = (p^2 - q^2 - a\gamma_1)^2 + (2pq - \alpha\gamma_2)^2 + \alpha\,(p^2 + q^2 + {}^1/_2\, r^2 + \alpha\gamma_1)$$
$$a = Px_c/C$$

When $\alpha = 0$, we obtain the following solutions:

a particular integral for the Delaunay case /10/

$$p^2 - q^2 - \alpha\gamma_1 = 0,\ 2pq - \alpha\gamma_2 = 0$$

a particular form of permanent rotations

$$q = r = \gamma_2 = \gamma_3 = 0,\ p = p_0 = \text{const},\ \gamma_1 = \pm 1$$

pendulum motions /11/

$$p = q = \gamma_3 = 0,\ r^{\cdot} = -\alpha\gamma_2,\ \gamma_1^{\cdot} = r\gamma_3,\ \gamma_2^{\cdot} = -r\gamma_1$$

permanent rotations

$$p = p_0 = \text{const},\quad q = 0,\quad \gamma_1 = \frac{p_0^2}{a},\quad \gamma_2 = 0,\quad = \frac{a\gamma_3^0}{p_0}$$
$$\gamma_3^{0 2} + \left(\frac{p_0^2}{a}\right)^2 = 1$$

pendulum motions

$$p = r = \gamma_2 = 0,\ q^{\cdot} = {}^1/_2\,\alpha\gamma_3,\ \gamma_1^{\cdot} = -q\gamma_3,\ \gamma_3^{\cdot} = q\gamma_1,\ q^2 + a\gamma_1^2 = 0$$

When $\alpha = -2\,(aC/l)^2$, $l = 2Cp\gamma_1 + 2Cq\gamma_2 + Cr\gamma_3$, we obtain a family of particular solutions corresponding to the intersection of the Kovalevskaya and Bobylev-Steklov cases /10/

$$p = p_0 = a\frac{C}{l},\quad q = 0,\quad r = \frac{l}{C}\gamma_3,\quad \gamma_1 = \frac{l^2}{2aC^2}(1-\gamma_3^2)$$
$$\gamma_1^{\cdot} = \frac{l}{C}\gamma_2\gamma_3,\quad \gamma_3^{\cdot} = -\frac{aC\gamma_2}{l_3}$$

Using the Euler angles

$$\psi^{\cdot} = \nu,\quad \varphi^{\cdot} = \nu\cos\theta,\quad \theta^{\cdot} = \frac{a}{2\nu}\cos\varphi,\quad \sin\varphi = 2\frac{\nu^2}{a}\sin\theta$$
$$\nu = \frac{l}{2C} = \text{const}$$

we can write the equations of motion in the form

$$\varphi^{\cdot 2} + \left(\frac{a}{2\nu}\right)^2 \sin^2\varphi = \nu^2,\qquad \theta^{\cdot 2} + \nu^2\sin^2\theta = \left(\frac{a}{2\nu}\right)^2$$

We see that if $(a/2\nu)^2 < \nu^2$, then the θ coordinate oscillates and the variation in φ corresponds to rotational motion. When $(a/2\nu)^2 > \nu^2$, then φ oscillates and θ "rotates". When $(a/2\nu)^2 = \nu^2$, both motions tend asymptotically to one of the permanent rotations of the body with the centre of mass lying on the vertical, while retaining their synchronization.

In the Goryachev-Chaplygin case $(y_c = z_c = 0,\ A = B = 4C;\ l = 4Cp\gamma_1 + 4Cq\gamma_2 + Cr\gamma_3 = 0)$ we take the generating integral in the form of a combination of the Goryachev-Chaplygin and energy integrals

$$\omega = r(p^2 + q^2) - ap\gamma_3 + \alpha(4p^2 + 4q^2 + r^2 + 2a\gamma_1),\quad a = Px_c/C$$

This yields a family of particular solutions /12/

$$p = 2\alpha\gamma_3({}^3/_2 - b\gamma_3^2),\ q = 2\alpha\gamma_2/\gamma_3,\ r = 8\alpha(b\gamma_3^2 - {}^1/_2)$$
$$\gamma_1 = -b\gamma_3^4 + {}^3/_2\,\gamma_3^2 - 1,\ b = a/32\alpha^2\ (\alpha \neq 0)$$

The quantity γ_3 is calculated by inverting the quadrature obtained from the equation

$$\gamma_3\gamma_3^{\cdot} = \pm 2\alpha\,[1 - \gamma_3^2 - (b\gamma_3^4 - {}^3/_2\gamma_3^2 + 1)^2]^{1/2}$$

We conclude by noting the following. A method of deriving stationary solutions from the system of equations $\partial\omega/\partial z_i = 0$, where ω is a bundle of known first integrals and z_i are phase variables, allied to the Levi-Civita method, is well-known. Use of the operators X_i^*, Y_i^* for this purpose shows two differences: 1) there is no need to include in the bundle the periodic integrals and integrals resulting from the redundancy of the variables; 2) particular solutions can be obtained from the integrals appearing on the degenerate levels of periodic integrals (sch as e.g. the Goryachev-Chaplygin integral).

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THEORY OF THE MOTION OF SYSTEMS WITH ROLLING*

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A mathematical model is proposed for describing the motions of a system with rolling, and with or without slippage. Conditions are given for the transition from one mode of motion to another. Examples are included.

Rolling without slippage is equivalent to determining a kinematic constraint, generally rheonomic /1/, described by differential equations linear with respect to the generalized velocities. The equations cannot usually be reduced to finite relations connecting the generalized coordinates, and therefore rolling without slippage represents a motion with a non-holonomic constraint. Study of the motion of a system with rolling, taking slippage into account, reduces to the study of the dynamics of a system with releasing kinematic constraints. Two problems arise in this context.

1°. Using differential equations to describe the motion of a system with rolling in the general case;

2°. Establishing the conditions for transferring from one rolling mode to another.

In the classical mechanics of non-holonomic systems where rolling without slippage is usually discussed, the second problem disappears, and the first problem was solved by Chaplygin, Voronets, Boltzmann, Hamel, et al. When a wheel with an elastic deformable type rolls without slippage, kinematic constraints appear which differ considerably from the classical non-holonomic constraints arising when a rigid body is rolling. The general equations of motion of a wheeled carriage executing small deviations from its uniform rectilinear motion were given in /2/, where the Keldysh theorem concerning the rolling motion of a wheel with an elastic tyre was used. The equations were generalized in /3/ to the case of the curvilinear motion of a wheeled carriage along a trajectory of fairly small curvature.

In general, the equations of motion of a system with rolling have the simplest form in the moving coordinate system /4, 5/ and must be written in the form of equations in quasi-coordinates. As we know, the equations of motion of a non-holonomic system are also written in this form /6/, therefore the equations in quasicoordinates are the most suitable for describing the motion of a system with rolling, with or without slippage. We must however generalize the well-known Boltzmann-Hamel equations to the case of a system with rheonomic kinematic constraints. The equations in quasicoordinates obtained in this manner solve the first of the above problems and can be used as a basis for the general theory of the motion of systems with rolling.

Investigation of the structural features of the phase space of a system with rolling also enables the second problem to be solved. It also becomes clear that the equations of kinematic constraints describing rolling without slippage can be regarded as the equations of some hypersurface Π in phase space. For the case of rolling without slippage we have the corresponding motion of a phase point along the surface Π in the region stable with respect to deviations from the surface Π. By determining the boundaries of this region we can solve the problem of the conditions governing the passage from rolling without slippage to rolling with slippage, and we can find the conditions for the reverse process to occur.

1. General equations of dynamics for a system with rolling.

Let the position of the system with rolling be defined by n generalized coordinates $q_1, q_2, \ldots, q_n$, and rolling without slippage by $n-m$ equations of the form

$$a_{ls}(q, t)\, q_s^{\cdot} + a_l(q, t) = 0 \quad (l = m+1, m+2, \ldots, n) \tag{1.1}$$

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